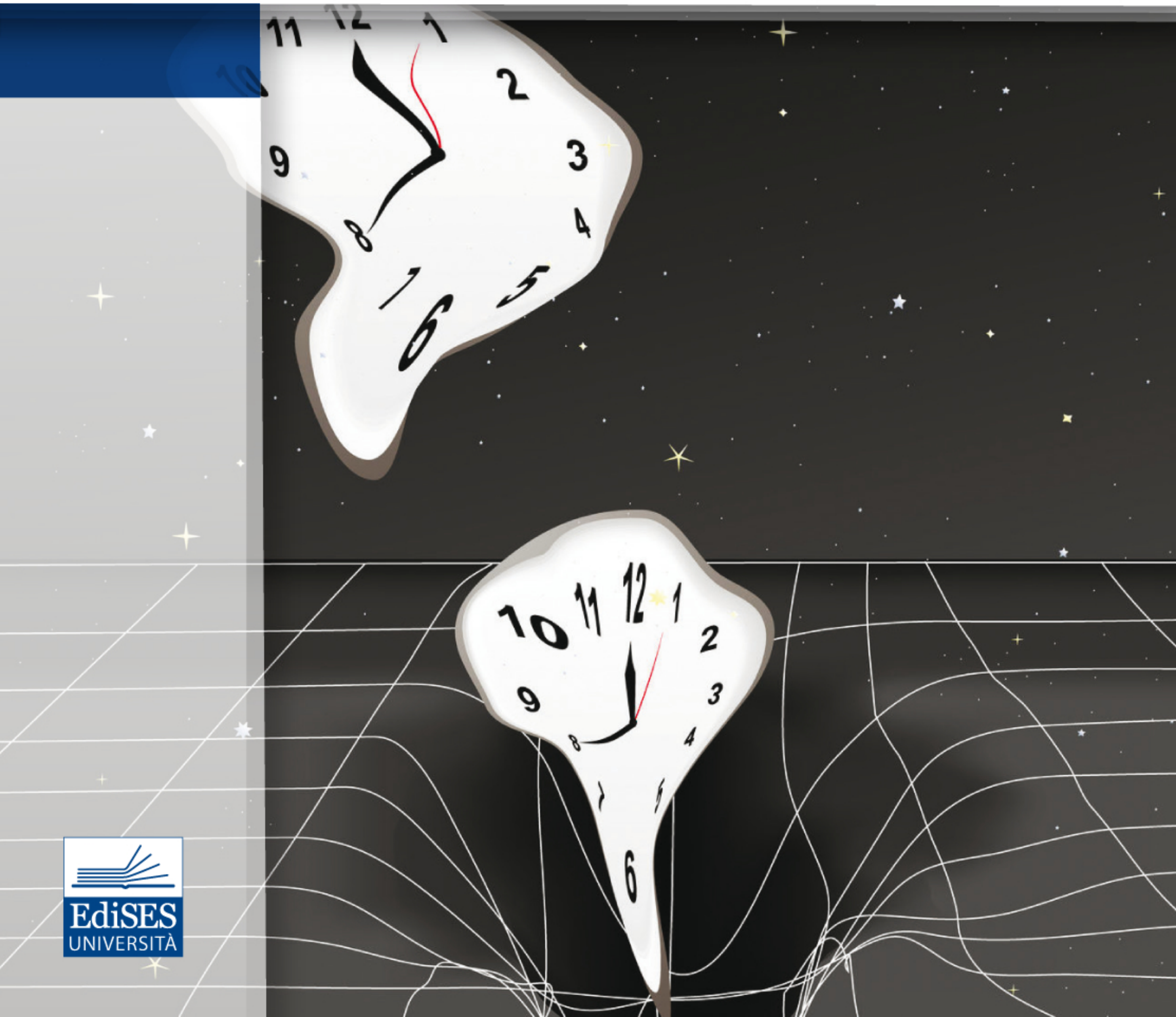
Francesco Bajardi

The Topological Invariant Approach

From Cosmology to Complex Systems



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9 8 7 6 5 4 3 2 1 0
2025 2024 2023 2022 2021

Le cifre sulla destra indicano il numero e l'anno dell'ultima ristampa effettuata

This book is based on the PhD thesis of Francesco Bajardi, written under the supervision of Prof. Carlo Altucci and Prof. Salvatore Capozziello.

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Stampato presso la
PrintSprint S.r.l. – Napoli

per conto della
Edises Edizioni S.r.l. – Piazza Dante Alighieri, 89 – Napoli

www.edisesuniversita.it
assistenza.edises.it

ISBN 978 88 3623 057 0

Preface

Topological invariants are usually defined as quantities which are preserved under homeomorphism transformations. This means that they are mathematical objects which do not depend on the local form of the spacetime, but only relies on its global structure, the topology. They are largely used in all physics branches, from gravitation up to complex systems, due to their capability of reducing the complexity of the dynamics and leading to exact solutions. This book is aimed at describing their applications in different contexts, such as modified theories of gravity, standard electromagnetism and biological systems. Regarding the former, it is well known that Einstein General Relativity is still considered the best accepted theory describing the gravitational interaction, but several shortcomings arise in the so called strong regimes. As a matter of facts, despite its success, General Relativity presents many unsolved issues and puzzles at any scales. Such problems can be partially solved by modified theories of gravity, which aim to extend the Einstein-Hilbert action to a more general one including other geometric terms. These latter can mimick the role of Dark Energy and Dark Matter, providing an effective energy-momentum tensor of the gravitational field. Among all the possible modifications of the starting actions, in this book modifications related to topological invariants are considered. Modified theories of gravity, often lead to higher-order field equations which cannot be analytically solved even in cosmological backgrounds. In this framework, reducing the order of the field equations, topological invariants can be particularly useful in order to find out exact solutions, well describing the today observations at the large scales. Moreover, as pointed out in the second part of the book, topological invariants can be used to construct gauge-invariant Lagrangians, which allow to fix the high-energy issues arising in the attempt of merging the formalism of Quantum Mechanics with that of General Relativity. In the second part of this work we will focus on a modified theory of gravity including a function of the Gauss–Bonnet topological surface term, showing that suitable field equations allow to find out exact solutions in a cosmological and in a spherically symmetric background.

The starting action is selected by means of the Noether Symmetry approach, a selection criterion aimed at finding theories containing symmetries. Modified $f(\mathcal{G})$ gravity, with $\mathcal{G}$ being the Gauss–Bonnet term, is studied in a D-dimensional spacetime, where higher-order cosmological and black holes solutions are provided. The Gauss–Bonnet scalar is, then, coupled to a dynamical scalar field, in order to make a comparison with the standard scalar-tensor theory of gravity. In all cases, General Relativity can be recovered as a particular limit.

In the third part of the book, we consider the Chern–Simons theory in odd dimensions. It is based on the Chern-Simons forms, whose exterior derivatives provide topological surface terms. This property make the theory *quasi*-Gauge invariants, namely invariant under gauge transformations up to a boundary term. We show that from very general and basic theories such as classical and quantum theories of gravity, Chern–Simons theory can lead to far beyond closely related fields to push concepts and applications to complex systems, there including the interactions between biomolecules, such as nucleic acids and proteins. Indeed, after providing cosmological and spherically symmetric solutions in D dimensions, we show that the theory can be also applied to biological systems. While applications to our Universe seems to be a straightforward consequence for a testbed of the theory itself, the use of Chern–Simons theory in understanding complex systems might look unusual and non-conventional.

Biological systems often exhibit complicated topological structures, such as nucleic acids or proteins, since different parts of the same molecule may assume a complicated three-dimensional shape (tertiary structure). When two or more tertiary structures interact, the resulting system fold into a quaternary structure, whose schematization represents one of the most controversial and discussed branch of science, due to the important implications in biology, microbiology, medicine *etc.* As an example, from the spatial configuration assumed by the DNA, it is possible to infer the place in which genomic mutations might occur, as well as the difference among phenotypes.

The link between Chern–Simons theories and the dynamics/interactions of complex biomolecules is the topological nature of the former which can be essential to describe the complicated physico-chemical and biological behavior of the latter, very much relying on their topology. Basically, the main idea is to describe the DNA curvature by using the same formalism adopted for the spacetime, treating the interactions among biological systems as driven by the same general principles that govern the gravitational interaction.

By merging the schematization approach lying behind the Chern–Simons theory with the more conventional ones coming from bioinformatic, it is possible to implement the nowadays knowledge of the biological scenario. In particular, the deterministic aspect of the former can be combined with bioinformatic techniques, which treat the biological issues from a stochastic point of view.

As a final remark, in light of the above mentioned applications, it is worth pointing out that this book can be understood as a first step towards the development of the so called "Topological Invariant Approach". More precisely, we aim to show that topological invariants can be considered in the framework of different fields to describe the corresponding dynamics, ranging from cosmology, black holes, up to complex systems. Throughout the history of physics it is possible to identify several approaches, developed to solve specific issues, but which subsequently spread out in different fields, because of their general validity. This is the case *e.g.* of symmetries, which nowadays play a fundamental role in almost all branches of science. Similarly, though topological invariants arose with the purpose of addressing evidences provided by the gravitational interaction, the same structure can be also applied to apparently unrelated fields. In particular, the vision on which topology, geometry and topological invariants are based on, is the key point of the approach. In this way, once addressing a configuration space to the given system, the evolution can be described under the same formalism as the space-time, so that the research for topological and geometrical features of the configuration space can provide information about the related dynamics. Therefore, the link with the gravitational interaction appears natural and straightforward, and the dynamical behavior of galaxies, stars and planets can be addressed to other different models.

To conclude, this book is organized as follows: in Chaps. 1 and 2 successes and shortcomings of General Relativity are outlined, and the main classes of modified theories of gravity are discussed. In Chaps. 3-8, different theories of gravity involving the Gauss–Bonnet scalar are studied, as well as scalar-tensor theories and non-local theories. Specifically, in Chap. 3, cosmological aspects such as energy conditions and slow–roll inflation are discussed in the framework of $f(\mathcal{G})$ gravity. The form of the starting action is selected by symmetry considerations, namely using the Noether symmetry approach. The prescription pursued to find out analytic cosmological solutions by Noether's approach is based on Ref. [47], while applications to early stages of the Universe and energy conditions on Ref. [146]. In Chap. 5 we find out exact solutions for $f(\mathcal{G})$ gravity in a spherically symmetric background, following Ref. [155]. In Chap. 6 we compare two classes of non-local integral kernel theories of Gauss–Bonnet gravity, outlining the main results of Ref. [158]. In Chap. 7 the equivalence between metric and affine scalar-tensor theories is discussed, remarking the differences and the common features. In particular, a function of the scalar field $f(\phi)$ is coupled to the scalar curvature (Sec. 7.1), to the torsion scalar (Sec. 7.2) and to the Gauss–Bonnet scalar (Sec. 7.3). For further details see Ref. [162]. The third part is devoted to basic foundations and applications of Chern–Simons theory. After outlining its main aspects in Chap. 9, in Chaps. 10 and 11 Chern–Simons gravity is applied to cosmology and spherical symmetry [177]. Finally, in Chaps 12 and 13 the applications to electromagnetism and biological system is respectively considered [225, 224]. With regards to this latter, in Sec. 13.1.1 the theory is applied to KRAS human gene, in order to study the effect of induced mutations to selected sequences. In Sec. 13.1.2 the same analysis is performed to SARS-COV 2 virus.

Keywords: Topological Invariants; Modified Theories of Gravity; Complex Systems.

"I managed to get a quick PhD
- though when I got it I knew
almost nothing about physics.
But I did learn one big thing:
that no one knows everything,
and you don't have to."
S. Weinberg

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Notation

We will set $\hbar = c = 8\pi G = 1$ unless otherwise indicated and we will use the following notation:

1. For the indexes:
   - Greek indexes $\{\alpha, \beta, \gamma ... = 0,1,2,3\} \rightarrow$ label the four dimensional curved space-time coordinates
   - Latin indexes $\{a, b, c ... = 0,1,2,3\} \rightarrow$ label the four dimensional flat space-time coordinates
   - Middle indexes $\{i, j, k ... = 1,2,3\} \rightarrow$ label the spatial coordinates
   - Symmetrization over the indexes will be indicated by the curly bracket, while anti-symmetrization by the square bracket
2. Let A_μ be a generic four-vector, we adopt the following:
   - $D_\nu A_\mu = A_{\mu;\nu}$ is the covariant derivative in terms of the Levi-Civita connection
   - $\partial_\nu A_\mu = A_{\mu,\nu}$ is the standard partial derivative
   - Christoffel connection will be indicated equivalently by $\Gamma^\alpha_{\beta\gamma}$ or $g^{\alpha\sigma}\{\sigma, \beta\gamma\}$
   - $\nabla_\nu A_\mu \rightarrow$ is the covariant derivative in terms of any connection except for the Levi-Civita connection.
3. We use the symbol $\mathscr{L}$ for Lagrangian density, while the Lagrangian will be denoted by $\mathcal{L}$.
4. For the Einstein tensor we use the notation $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$
5. The derivative with respect to the variable will be indicated by the subscript variable or sometimes by the subscript variable in the partial derivative

6. $\Box$ stands for the four-dimensional D'Alembert operator $\Box = g_{\mu\nu}\nabla^{\mu}\nabla^{\nu}$
7. $\mathcal{X}$ represents the generator of a certain symmetry, while $X = \mathcal{X} + \dot{\eta}^i \partial_{\dot{q}^i}$ is the Noether vector

The metric signature adopted is $(+, -, -, -)$.

We will introduce less important symbols during construction.

List of Acronyms

- ADM: *Arnowitt-Deser-Misner*
- CFT: *Conformal Field Theory*
- AdS: *Anti de Sitter*
- DEC: *Dominant Energy Condition*
- FLRW: *Friedmann-Lemaitre-Robertson-Walker*
- GR: *General Relativity*
- GW: *Gravitational Wave*
- IDGs: *Infinite Derivative Theories of Gravity*
- IKGs: *Integral Kernel Theories of Gravity*
- IR: *Infrared Light*
- NEC: *Null Energy Condition*
- PN: *Post-Newtonian*
- QFT: *Quantum Field Theory*
- RBD: *Receptor Binding Domain*
- SEC: *Strong Energy Condition*
- STEGR: *Symmetric Teleparallel Equivalent of General Relativity*
- TEGR: *Teleparallel Equivalent of General Relativity*
- UV: *Ultraviolet Light*
- WDW: *Wheeler-DeWitt*
- WEC: *Weak Energy Condition*

To my grandmother, Maria Teresa

Part I

INTRODUCTION AND PRELIMINARIES

1

Overview of General Relativity: Successes and Shortcomings

[1] The Hilbert-Einstein action, linear in the Ricci curvature scalar R, gives rise to the field equations of General Relativity (GR), which is the theory of gravity capable of fitting a huge amount of phenomena ranging from gravitational waves (GWs), astrophysical compact objects, black holes up to cosmology. At the astrophysical scales, GR soon obtained a great success after the observations of the light deflection, followed by the Radar Echo Delay and the exact estimation of the precession of the perihelion of Mercury in its orbit around the sun. The above mentioned successes come from the application of the theory to a spherically symmetric space-time of the form

$$ds^2 = P(r,t)^2 dt^2 - Q(r,t)^2 dr^2 - r^2 d\Omega^2, \tag{1.1}$$

with Ω being the two-sphere defined as $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$. Once replacing the interval (1.1) in the Einstein field equations, it turns out that the only solution is

$$ds^2 = \left(1 - \frac{r_S}{r}\right) dt^2 - \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 - r^2(sin^2\theta d\phi^2 + d\theta^2), \tag{1.2}$$

which is static and contains two intrinsic singularities. One of them is an intrinsic divergence occurring for $r = 0$, due to the curvature generated by the compact object, at the

[1] In this Chapter we consider the Newton constant G_N, subsequently set to $1/8\pi$.

center of which any information is missed. The other singularity occurs when the radius is equal to the so called "Schwarzschild radius" r_S, defined as

$$r_S = 2G_N M, \tag{1.3}$$

with G_N being the Newton coupling constant and M the mass of the compact object. It can be shown that this latter singularity is coordinate-dependent, and can be deleted by means of an appropriate transformation (Kruskal-Szekeres coordinates). The plane $r = r_S$ is the "Event Horizon" and can be interpreted as the boundary beyond which events cannot affect an observer. The recent black hole image at the center of M87 galaxy, showed that these theoretical predictions are consistent with experimental observations [1].

The application of GR to homogeneous and isotropic space-times led to better understand the cosmological evolution crossed by the Universe, from the Big Bang to the Dust Matter Dominated Era. Using a cosmological perfect fluid with equation of state $p = \gamma\rho$, the Einstein field equations provide the solution

$$\frac{a}{a_0} = \left(\frac{t}{t_0}\right)^{\frac{2}{3(\gamma+1)}} \qquad \rho(t) = \left[6\pi G_N(1+\gamma^2)t^2\right]^{-1}, \tag{1.4}$$

where a spatially–flat Friedmann–Lemaitre–Robertson–Walker (FLRW) universe of the form

$$ds^2 = dt^2 - a(t)^2[dx_1^2 + dx_2^2 + dx_3^2], \tag{1.5}$$

must be considered to obtain Eq. (1.4). Depending on the value of γ, three different epochs can be identified:

- $\gamma = \frac{1}{3} \rightarrow$ Radiation fluids
- $\gamma = 1 \rightarrow$ Stiff matter fluids
- $\gamma = 0 \rightarrow$ Dust matter fluids

Experimental observations confirm that the evolution of the Universe went through different epochs, predicted by GR cosmology with high precision.

At the astrophysical scales, linearized Einstein field equations show that GR admits the presence of GWs propagating outward from their source at the speed of light. Specifically, considering a small perturbation $h_{\mu\nu}$ of the Minkowski flat metric tensor

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1\,, \tag{1.6}$$

a D'Alembert equation of the form

$$\Box h^{\mu}_{\nu} = -2\left(\mathcal{T}^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}\mathcal{T}\right)\,, \tag{1.7}$$

[2] can be obtained from the field equations, where $\mathcal{T}$ is the trace of the energy–momentum tensor $\mathcal{T}^{\mu}_{\nu}$. In vacuum the above equation describes propagating waves at the speed of light. Using the TT gauge condition, the general solution reads:

$$\begin{gathered} h_{\mu\nu} = e^{+}_{\mu\nu}h_{+} + e^{\times}_{\mu\nu}h_{\times}\,, \\ e^{+}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad e^{\times}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{gathered} \tag{1.8}$$

According to the standard model, GWs in GR are described by spin-2 massless particles, with spin orientated in the same direction of motion.

For many years, GWs represented only a theoretical solution of field equations. In 2015, the Laser Interferometer Gravitational-Wave Observatory (LIGO) revealed a GW event (GW150914) and opened a new window in astrophysics and cosmology [2].

The GW production occurred during the merging of two black holes with masses of 29 $M_{\odot}$ and 36 $M_{\odot}$. The merging process produced a black hole of 62 $M_{\odot}$. The remaining (3 $M_{\odot}$) mass-energy was released in form of gravitational radiation. The observation gave a

[2] The operator $\Box$ is the D'Alembert operator defined as $D_{\mu}D^{\mu}$, with D_{μ} being the covariant derivative

double result: confirmed the existence of GWs and of stellar mass black holes.
After this first detection, several other events have been observed thanks to the LIGO-VIRGO collaboration, and further detections are expected in the forthcoming years.
When the cosmological constant is considered and dominating, the vacuum solution of the Einstein field equations in a FLRW space-time provides a scale factor of the form

$$a(t) = a_0 e^{\sqrt{\frac{\Lambda}{3}}\,t}, \tag{1.9}$$

denoting an exponentially accelerated universe.
The cosmological constant was introduced to explain the today observed accelerated cosmic expansion, physically interpreted as a form of energy which should represent the 68% of the Universe, called *Dark Energy.* The today accepted formulation of gravity, includes the cosmological constant as a fundamental component in the Einstein field equations, which therefore reads as:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G_N\, \mathcal{T}_{\mu\nu}, \tag{1.10}$$

where $R_{\mu\nu}$ is the Ricci tensor, R the Ricci scalar, $g_{\mu\nu}$ the metric tensor and $\mathcal{T}_{\mu\nu}$ the energy–momentum tensor of matter fields.
Those mentioned above are only a few part of the results gained by GR during more than one hundred years. In spite of all this, it also provided some results which disagree with experiments. For instance GR is not able to predict the right correlation between mass and radius of compact objects. Another example is given by the speed of the farest stars orbiting around the center of a given galaxy, which is experimentally lower than theoretically expected (see galaxy rotation curve problem [3]). To theoretically fix this issue, the missing matter was addressed to a fluid with zero pressure, called *Dark Matter*. It is supposed to represent the 26.8% of the Universe but has never been observed directly.

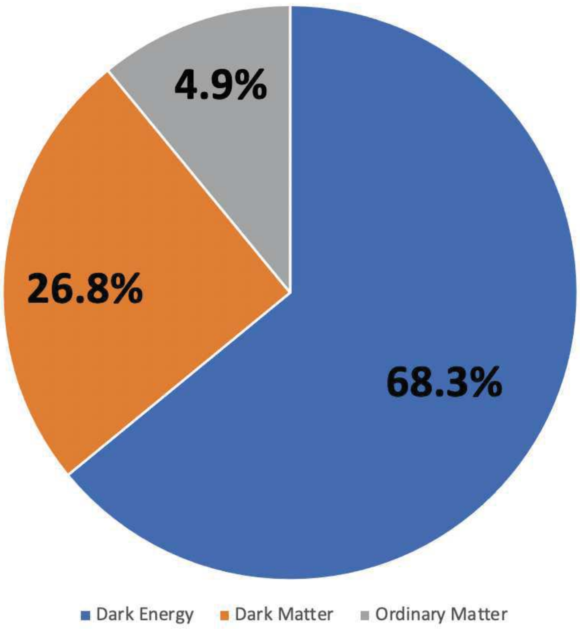

Figure 1: *Energy–matter content in the Universe.*

As the accelerated expansion cannot be predicted by GR without invoking the presence of Dark Energy, the galaxy rotation curve cannot be fitted without Dark Matter; however, even considering the cosmological constant, at the quantum level there is a discrepancy of 120 order of magnitude between the theoretically predicted value and the experimentally calculated one.

On the other hand, the "local" formulation of GR seems completely in disagreement with the intrinsic "non-locality" of quantum mechanics.

Quantum mechanics was the most revolutionary theory of the last century, which opened the doors to a completely new vision of physics at the high energy. The determinism of classical mechanics was replaced by a probabilistic interpretation of small-scale phenomena, which seemed to be the only way to fit all the experimental results. As we gained a theory capable of describing almost all the evidences provided by the quantum world, we lost the capability to exactly predict the time evolution of the system. Soon after, Quantum Field Theory (QFT) arose with the purpose to describe all the fundamental interactions under the same standard. It was soon clear that this prescription could not be applied to the gravitational interaction. Indeed, as quantum mechanics is probabilistic by nature, gravity is in turn described by Einstein's GR, where non-local interactions are not allowed. So far, a theory capable of describing both the large-scale structure and the Ultraviolet (UV) scale results is still missing. Moreover, neither QFT nor GR hold at the Planck scale, where a new physics is probably needed. On the one hand, despite all the experimental confirmations of quantum mechanics, we still miss its deep meaning; on the other hand, although GR is mathematically consistent and well developed, it presents some inconsistencies even at the large-scales. Any attempt to merge the formalism of GR with that of QFT have failed. Even though QFT in curved space-time addressed several evidences provided by the small-scales observations (such as Hawking Radiation, Unruh effect or cosmic inflation) it suffers several shortcomings. Indeed, it turns out that GR can be renormalized up to the second loop level [4], which means that incurable divergences arise once adapting the same scheme as QFT to gravity. In addiction, unlike the other fundamental interactions, GR cannot be treated under a Yang-Mills formalism, due to the lack of a Hilbert space and a probabilistic interpretation of the wave function. For these reasons, a coherent and self-consistent theory of quantum gravity is one of the most studied topic nowadays [5, 6, 7, 8, 9, 10, 11]. In the last few years, the quantum formalism was adapted to cosmology, where the dynamics can be reduced considering a minisuperspace of the variables. It represents a "toy model" which does not claim to be complete, but yields several important results in the understanding of the early-stage of our Universe [12, 13, 14, 15, 16].

2

Modified Theories of Gravity

Before introducing the main classes of modified theories of gravity it is useful to overview the state of art of GR modifications and the reasons why extending gravity. As mentioned above, while the electroweak and the strong interaction are Yang-Mills gauge theories, GR is invariant under diffeomorphism transformations, which involve coordinates instead of fields. Moreover, according to the geometric description of gravity, in view of a possible quantum scheme, the space-time metric should represent both a dynamical field and the background; this is not the case of other interactions, whose treatment is simplified by the assumption that the space-time is supposed to be flat. In 1988 Lasenby, Doran and Gull proposed to deal with the flat tangent space of the Riemannian manifold, treating GR as a gauge theory with respect to the local Lorentz group [17]; in order to pass from the curved to the flat space-time, a mathematical tool called *tetrad fields* is necessary, which in turn becomes the fundamental dynamical field. The formalism adopted is the so called Einstein–Cartan formalism [18, 19, 20, 21], where the connection is generally independent of the metric and the two-form curvature must be found through Cartan's structure equations [18, 22]. This implies that curvature can be used along with torsion to simultaneously label the space-time, so that the theory reduces to standard GR as soon as anti-symmetric part of the connection vanishes. This approach is not aimed at solving all the problems occurring in GR at the small scales regime, since even under an Einstein-Cartan formalism, several shortcomings are still suffered. As an example, neglecting the asymptotic safety scenario [23, 24, 25, 26], the theory can be renormalized only up to the one-loop level. At the large scales, early and late-time Universe acceleration cannot be predicted without introducing Dark Energy, as well as the galaxy rotation curve cannot be fitted without Dark Matter. In this framework, modified theories of gravity arose with the purpose of solving such shortcomings, by taking into account alternatives to the Hilbert–Einstein action. In the first instance, they can be distinguished in two main categories: purely metric theories and metric-affine theories. The former (which will be the main focus of this book) admits the metric tensor as the only fundamental filed. The latter disentangles the contribution of the metric from the affine connection, such that no relations between $\Gamma^{\alpha}_{\mu\nu}$ and $g_{\mu\nu}$ occur. This prescription is usually called "Palatini formalism" (see [27, 28, 29, 30] for basic foundations and applications). One of the most famous extensions of GR is the $f(R)$ gravity, which introduces into the action a function of the scalar curvature. Similarly, the $f(T)$ gravity considers a function of the so called *torsion scalar* into the action. However, the Hilbert–Einstein Lagrangian can be extended in several ways, such as introducing the coupling between geometry and scalar fields, higher-than-fourth order terms involving the D'Alembert operator $\Box^n$, or higher-order curvature invariants (as well as $R^{\mu\nu}R_{\mu\nu}$ or $R^{\mu\nu p\sigma}R_{\mu\nu p\sigma}$). All these theories can be treated either with respect to the purely metric or to the Palatini formalism. In this book, we assume the affine connection to be linked to the metric tensor, such that the corresponding field equation solutions can be uniquely determined by knowing the space-time line element. Specifically, we will focus on those modified theories of gravity somehow related to topological invariants, such us modified Gauss–Bonnet gravity and Chern–Simons gravity. An exhaustive treatment regarding other modified theories of gravity can be found in [31]. For specific discussions see *e.g.* [32, 33, 34, 35, 36] for $f(R)$ gravity, [37, 38, 39, 40, 41, 42] for $f(T)$ gravity, [43, 44, 45, 46] for scalar-tensor gravity, [47, 48, 49] for actions depending on second-order curvature scalars. Lagrangians of most modified theories of gravity contain unknown functions which cannot be directly constrained by experimental observations. Therefore, it comes natural wondering how to select the shape of the function among all possible choices.

One possible remedy, largely considered in the literature (see App. B), is to use a selection criterion aimed at finding actions containing symmetries. It is called *Noether Symmetry Approach* and will be particularly used in the second part of this work to select modified Gauss–Bonnet theories with symmetries. Specifically, as better pointed out in App. B, Noether theorem can be used as an approach to reduce the dynamics and find out exact analytic solutions of the field equations in modified gravity.

2.1 Brief Introduction on Modified Theories of Gravity

Modified theories of gravity, in some context, are capable of fixing GR inconsistencies, at infrared (IR) and UV scales. GR can be modified in several ways, depending on the scale and on the theoretical issues considered. As a matter of fact, GR does not account for the most general classical theory of gravity, but it relies on several assumptions. Most of them are motivated neither by experimental observations nor by strong theoretical reasons, but was introduced with the aim to construct a suitable theory leading to analytic solutions. It is beyond any discussion that the description of the gravitational interaction through the space-time geometry was perhaps the greatest intuition of XX century, and the consequent approval marked a turning point in the physical comprehension of phenomena. However, in order to gain such a predictive power and to obtain analytic results, many hypothesis was adopted; in what follows we analyze the main assumptions lying behind GR.

- Equivalence Principle and Symmetric Connection.

Let us first consider the assumption of symmetric connection, based on the requirement for the validity of the Equivalence Principle. The Weak Equivalence Principle affirms that there is no difference between gravitational field and accelerated systems, so that a free falling reference frame is completely equivalent to a system with no gravitational field. In other words, according to the weak Equivalence Principle, it is always possible to locally link the curved space-time to a flat tangent Minkowski space-time. The equivalence between the gravitational and the intertial mass is then automatically implied, and

nowadays several experiments confirm such equivalence with a precision of 1 part over 10^{14}. Despite this, it is just an assumption motivated by macroscopic observations, which surely holds at large scales, though it is still unclear whether it keeps being valid at Planck scales. In order to treat the gravitational interaction under the same standard as the other fundamental interactions and to construct a coherent theory of quantum gravity, such a precision of 10^{-14} makes a crucial difference, since admits the possibility that at higher scales the ratio m_g/m_I might pull rapidly away from 1.
As usual in GR, the form of the Christoffel connection can be found by imposing the "metricity condition" $D_\alpha g_{\mu\nu} = 0$, by means of which the following identity

$$D_\alpha g_{\mu\nu} + D_\nu g_{\alpha\mu} - D_\mu g_{\nu\alpha} = 0, \tag{2.1}$$

must hold. Considering the definition of the covariant derivative and assuming $\Gamma^\alpha_{[\mu\nu]} = 0$, it turns out that the only possible connection in GR is the Levi–Civita connection, that is[1]:

$$\hat{\Gamma}^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\alpha p} \left(\partial_\mu g_{p\nu} + \partial_\nu g_{\mu p} - \partial_p g_{\mu\nu} \right). \tag{2.2}$$

However, once the metricity condition and the hypothesis of symmetric connection are relaxed, the same computation leads to a more general form of $\Gamma^\alpha_{\mu\nu}$, comprehending other non-trivial terms. It reads [50, 51, 52]:

$$\Gamma^\alpha{}_{\mu\nu} = \hat{\Gamma}^\alpha{}_{\mu\nu} + \frac{1}{2} g^{\alpha\lambda} \Big(T_{\mu\lambda\nu} + T_{\nu\lambda\mu} + T_{\lambda\mu\nu} \Big) + \frac{1}{2} g^{\alpha\lambda} \Big(-Q_{\mu\nu\lambda} - Q_{\nu\mu\lambda} + Q_{\lambda\mu\nu} \Big), \tag{2.3}$$

where $\hat{\Gamma}^\alpha{}_{\mu\nu}$ denotes the Levi-Civita connection and $Q_{\beta\mu\nu}$, $T^\alpha{}_{\mu\nu}$ are rank-three tensors defined as:

$$Q_{\beta\mu\nu} = \nabla_\beta g_{\mu\nu} \qquad T^\alpha_{\mu\nu} = 2\Gamma^\alpha_{[\mu\nu]}. \tag{2.4}$$

The formalism in which the metric is disentangled from the connection, so that this latter is no longer "metric compatible", is known as *Einstein-Cartan-Sciama-Kibble* formalism.

[1] Hereafter in this chapter, the Levi-Civita connection will be denoted by a hat on the top.

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ISBN - 978-88-3623-057-0
9 788836 230570